Simple Z_2 lattice gauge theories with fermionic matter at finite density

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Chaos, Duality & Topology in CMT
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C. Prosko, S.-P. Lee, and JM, PRB 96, 205104 (2017) [arXiv:1708.08507]

Outline

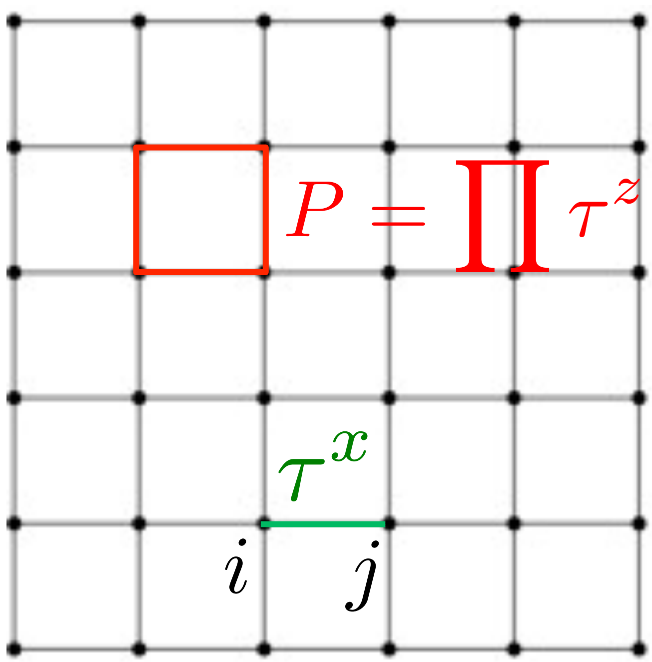
- A brief review of Z_2 gauge theory with matter
- Constrained Z_2 gauge theories with fermionic matter
- Unconstrained Z_2 gauge theories with fermionic matter
- Conclusion and outlook

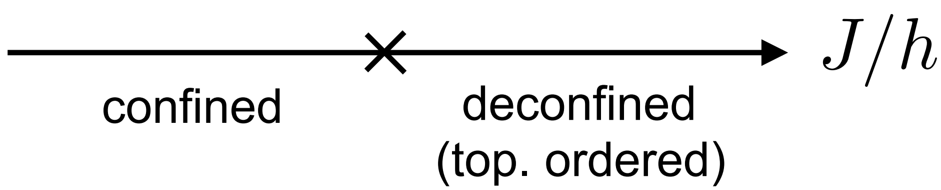
Z_2 gauge theory

- Introduced by Wegner (1971) as a model of continuous phase transition without a local order parameter

$$H_g = -J \sum_i P_i - h \sum_{\langle ij \rangle} \tau^x_{ij}$$

$$\sim \int \left(\mathbf{B}^2 + \mathbf{E}^2 \right)$$



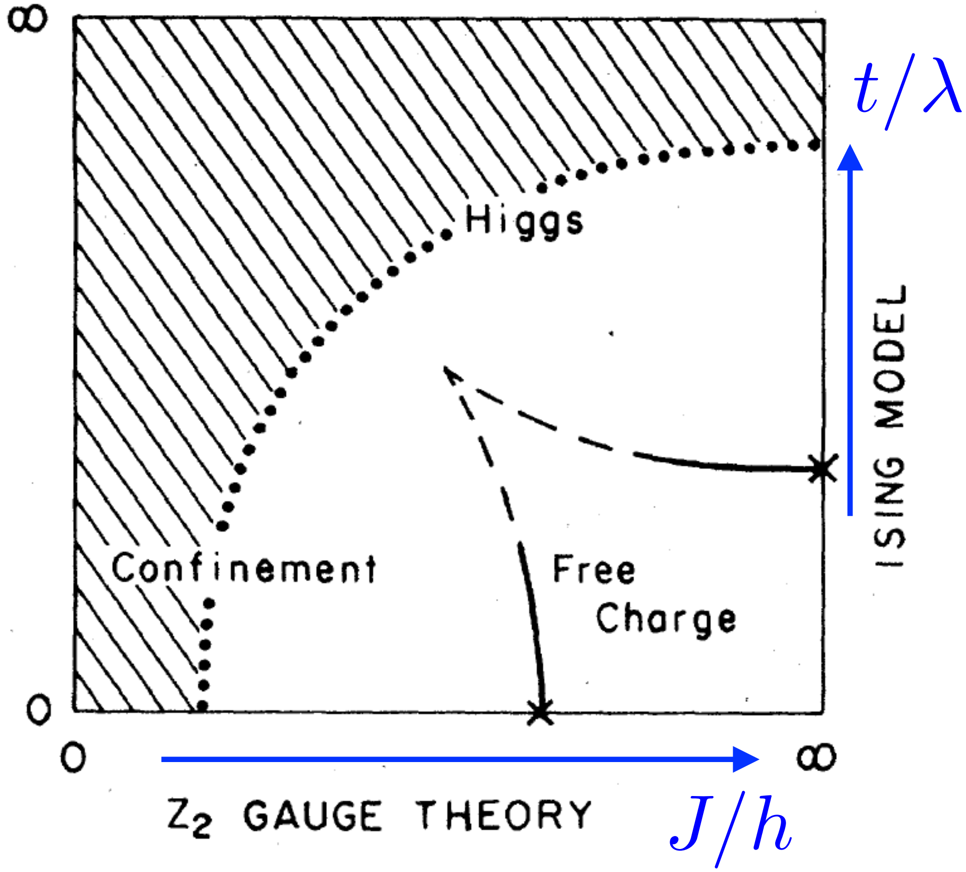


Z_2 gauge theory with matter

- In CMP, Z_2 gauge fields are **emergent** (slave-particle approaches) and are always accompanied by **dynamical matter fields**
- E.g.: Z_2 spin liquid = Z_2 gauge field + fermionic/bosonic spinons
- If matter is gapped, can be integrated out → reduces to pure gauge theory (e.g., topological phases)
- If matter is gapless, cannot be integrated out! Strongly coupled gauge + matter degrees of freedom, as hard to solve as original formulation (or harder).

Z_2 gauge theory with bosonic matter

$$H = -J\sum_i P_i - h\sum_{\langle ij\rangle}\tau^x_{ij} - t\sum_{\langle ij\rangle} s^x_i \tau^z_{ij} s^x_j - \lambda \sum_i s^z_i$$



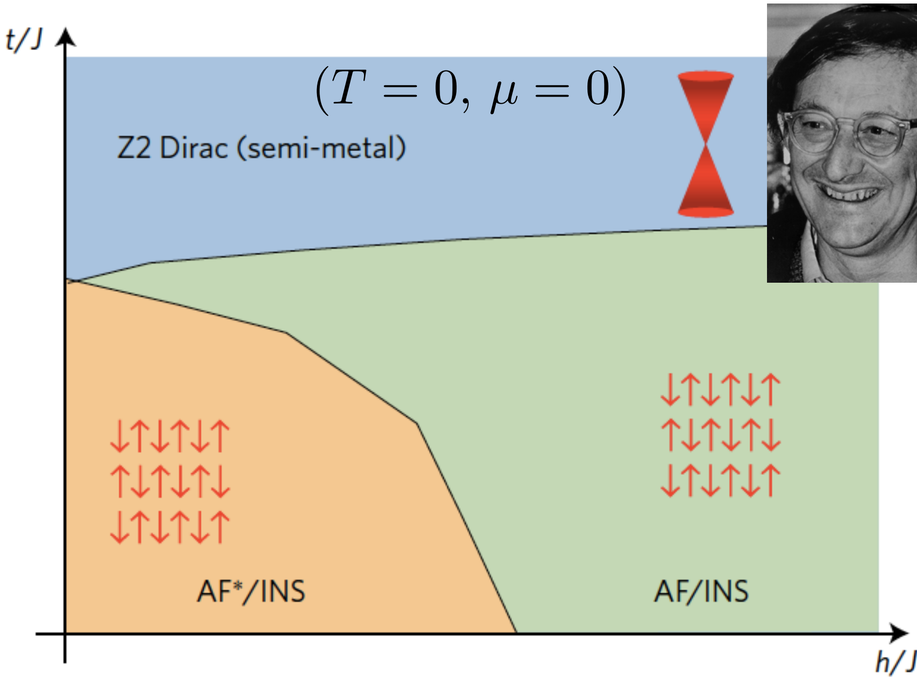
Fradkin, Shenker, PRD '79

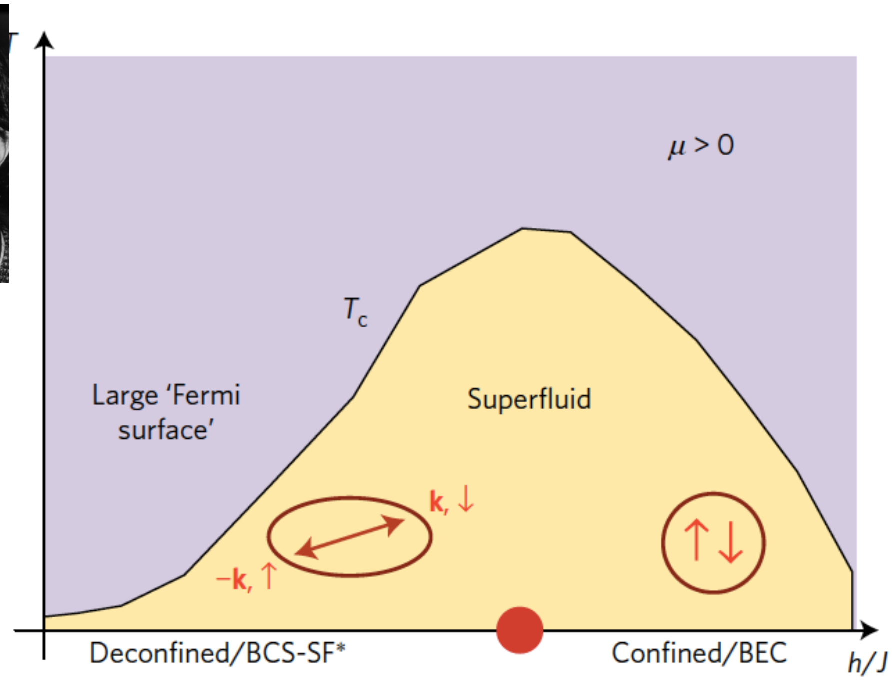
- Z_2 gauge field + Ising matter field
- Matter has qualitative effect: phase diagram not just "tensor product" of Z_2 gauge theory and Ising model

Z_2 gauge theory with fermionic matter

- Gauge theories with gapless **fermionic** matter have been much less studied; QMC simulations of such theories typically have a **sign problem**

- Exception: Z_2 gauge field + spinful fermions has no sign problem! (Gazit et al., Nat. Phys. '17; Assaad, Grover, PRX '16) QMC simulations at & away from half-filling, T=0 and T>0 (t: fermion n.n. hopping)





Gazit et al., Nat. Phys. '17

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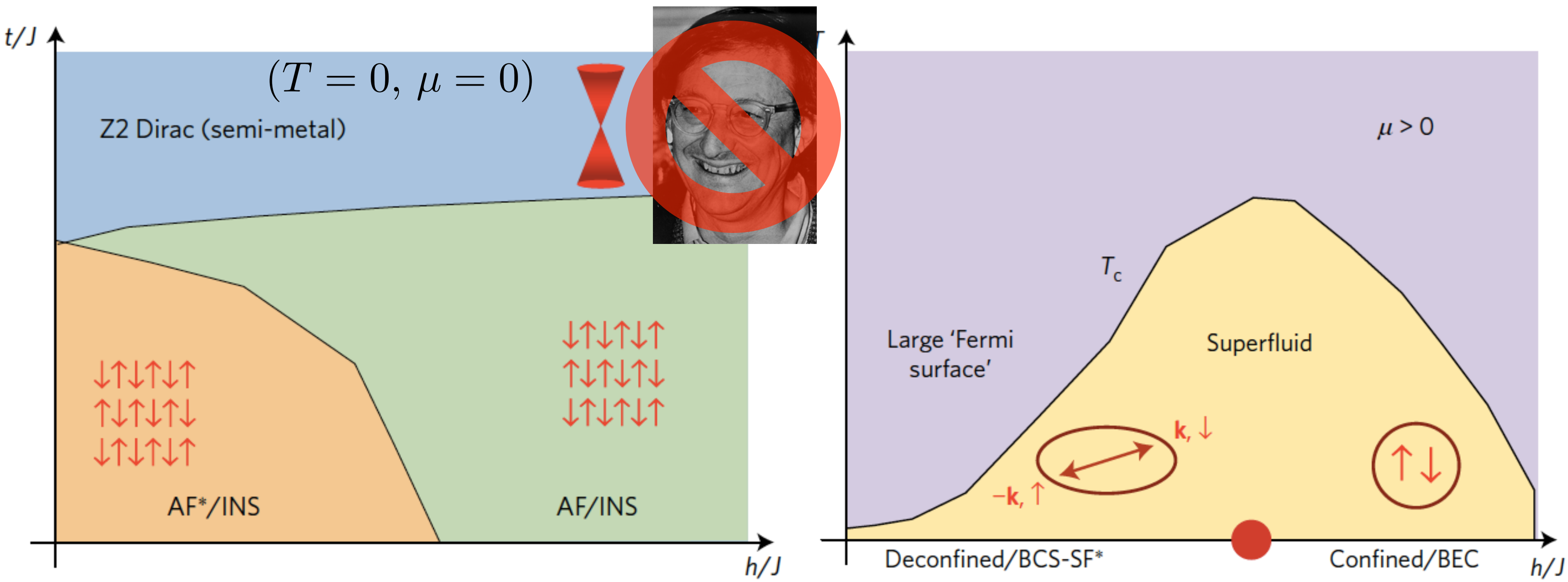

Gazit et al., Nat. Phys. '17

Z_2 gauge theory with fermionic matter

- Exactly/easily soluble Z_2 gauge theories with fermionic matter?
- Spinless fermions? (would have a sign problem)
- Majorana fermions?
- Two main types of theories:
  - **Constrained**: Gauss' law is imposed = zero background Z_2 charge (e.g., Gazit et al.)
  - **Unconstrained**: Gauss' law is not imposed = all background Z_2 charge sectors are kept in the physical Hilbert space (e.g., Assaad & Grover) = theory with a $(Z_2)^N$ symmetry with N = # of sites.

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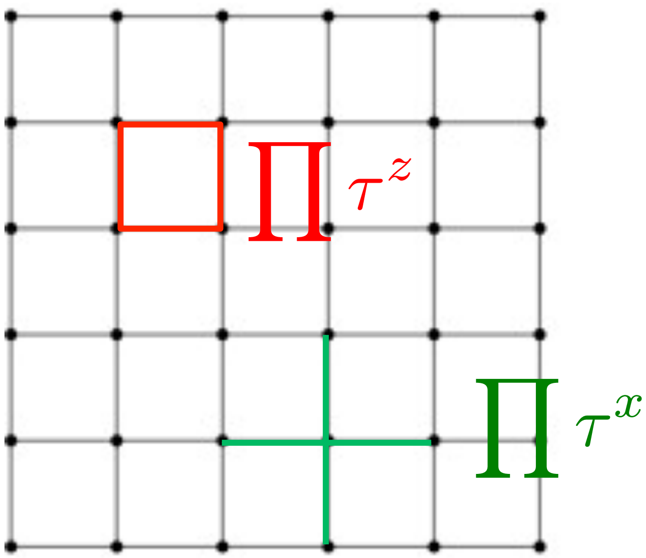
Wegner vs Kitaev

- The choice of Z_2 gauge theory Hamiltonian is not unique, as long as it is gauge invariant

- Kitaev's **toric code** is another choice:

$$H_g = -J \sum_{\square} \prod_{ij \in \square} \tau^z_{ij} - h \sum_{+} \prod_{ij \in +} \tau^x_{ij}$$

$$\sim \int \left(\mathbf{B}^2 + (\nabla \cdot \mathbf{E})^2 \right)$$



- Only electric field gradients cost energy = forces deconfinement of Z_2 electric charges (electric flux lines cost no energy)

Toric code + spinless fermions

- Consider Kitaev's Z_2 gauge theory coupled to spinless fermions. For simplicity set magnetic field term to zero (J = 0)

$$H = -t \sum_{\langle ij \rangle} c_i^\dagger \tau_{ij}^z c_j - \mu \sum_i c_i^\dagger c_i - h \sum_{+} \prod_{ij \in +} \tau_{ij}^x$$

makes τ^z dynamical

- Gauge transformation operator:

$$G_i = (-1)^{c_i^\dagger c_i} \prod_{ij \in +_i} \tau_{ij}^x, \qquad [H, G_i] = 0, \ \forall i$$

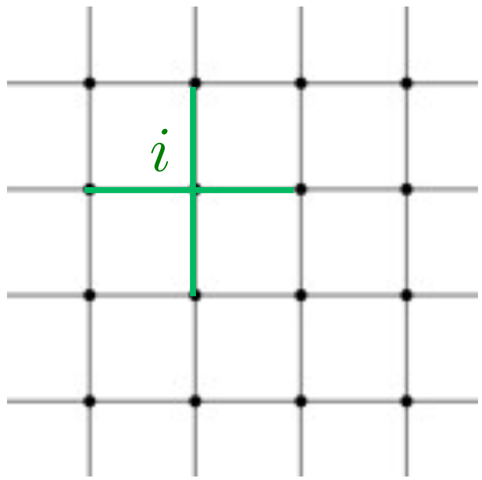
- Constrained gauge theory: must impose Gauss' law, $G_i = 1, \ \forall i$

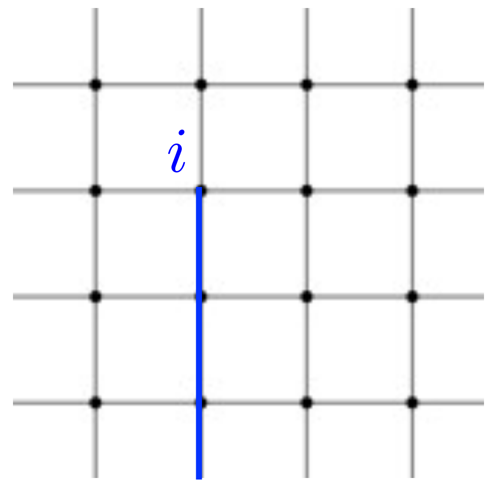
- Would have a sign problem in QMC (spinless)

Duality mapping

- To solve this model, introduce disorder (dual) variables:

$$\sigma_i^z = \prod_{ij \in +_i} \tau_{ij}^x \qquad \sigma_i^x = \prod_{n \geq 0} \tau^z_{i-(n+1)\hat{y}, i-n\hat{y}}$$





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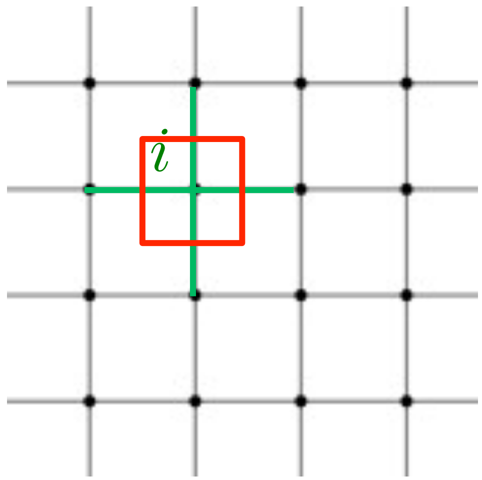

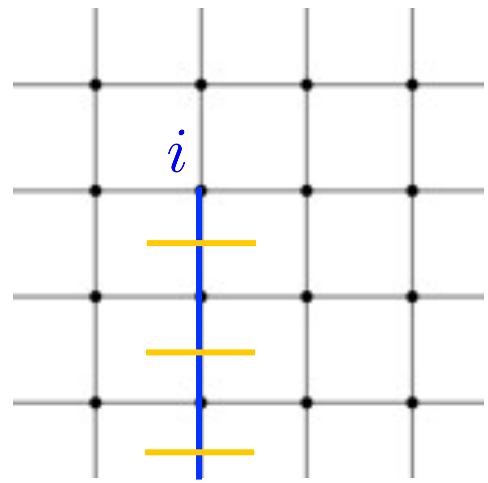

- "Electric-magnetic" dual of the usual disorder variables on the dual lattice (Kogut, RMP '79)

Duality mapping

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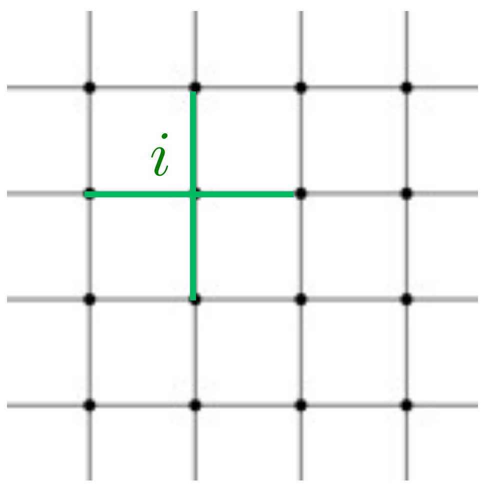
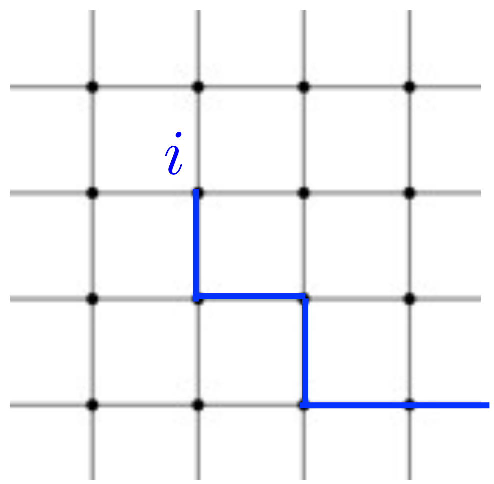




- Choice of string operator is arbitrary

Duality mapping

- In a given flux sector: $\tau^z_{ij} = B_{ij}\sigma^x_i\sigma^x_j$

  B_{ij} is a **static** Z_2 gauge field with the same flux as τ^z.

- In the dual variables, the Hamiltonian is

$$H = -t\sum_{\langle ij\rangle} B_{ij}\sigma^x_i\sigma^x_j c^\dagger_i c_j - \mu\sum_i c^\dagger_i c_i - h\sum_i \sigma^z_i$$

- Impose Gauss' law constraint:

$$G_i = (-1)^{c^\dagger_i c_i}\sigma^z_i = 1 \Rightarrow \sigma^z_i = 1 - 2c^\dagger_i c_i$$

- Introduce new **gauge-invariant** fermion operators:

$$\tilde{c}_i = \sigma^x_i c_i,\ \tilde{c}^\dagger_i = \sigma^x_i c^\dagger_i$$

Emergent fermions

- Maps to **free gauge-invariant fermions** in a **static** Z_2 gauge field:

$$\tilde{H} = -t \sum_{\langle ij \rangle} B_{ij} \tilde{c}_i^\dagger \tilde{c}_j - (\mu - 2h) \sum_i \tilde{c}_i^\dagger \tilde{c}_i$$

- Gauge coupling *h* simply shifts chemical potential!

- For $h = \mu/2$ (half-filling), exact ground state is the **π-flux phase** (Lieb, PRL '94)

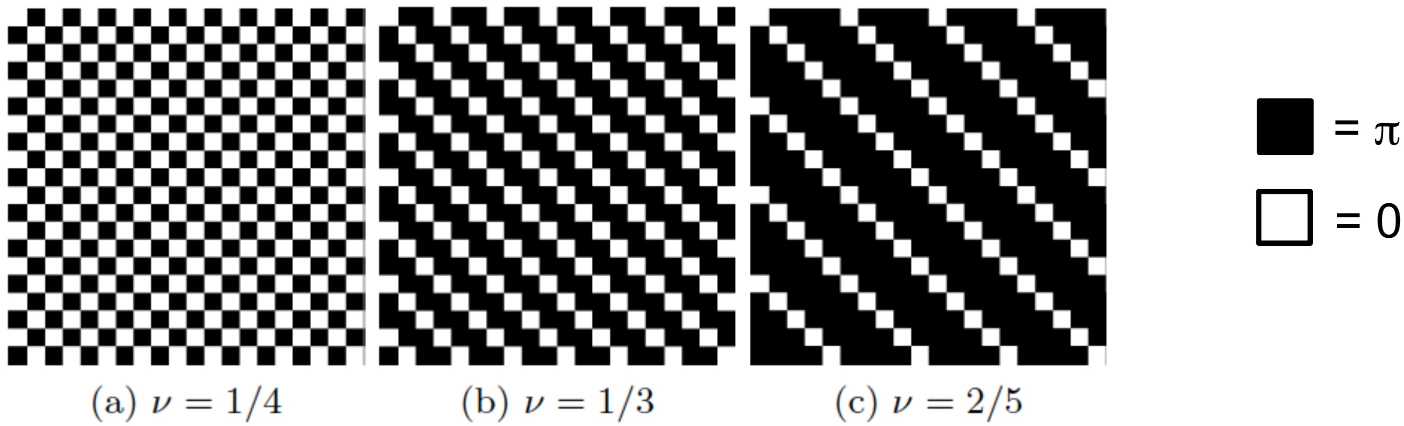
- Emergent Dirac fermions without breaking of physical translation symmetry

- Obeys **modified Luttinger relation** for Z_2 fractionalized phases (Paramekanti & Vishwanath, PRB '04)

$$\nu = \frac{1}{2} + \frac{A_{\text{FS}}}{(2\pi)^2} \mod \mathbb{Z}$$

Z_2 flux crystals

- Other rational p/q fillings: can do “classical” MC (flux optimization)



- Spontaneous translational symmetry breaking: **flux crystals** with $2\pi p/q$ flux per plaquette on average

- For U(1) fluxes optimal configuration is uniform $2\pi p/q$ flux per plaquette (Hasegawa et al., PRL ‘89); for Z_2 fluxes crystalline order is necessary

- **Semimetallic** states with two emergent Dirac cones in (reduced) BZ; Z_2 violation of Luttinger’s theorem for even-denominator fillings

Spinful fermions

$$H = -t \sum_{\langle ij \rangle, \sigma} c_{i\sigma}^\dagger \tau_{ij}^z c_{j\sigma} - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} - h \sum_{+} \prod_{ij \in +} \tau_{ij}^x$$

- Additional SU(2) spin symmetry; at half-filling U(1) charge is enlarged to SU(2) pseudospin
- Gauge transformation operator: $G_i = (-1)^{\sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}} \prod_{ij \in +_i} \tau_{ij}^x$
- Introduce disorder variables as before
- Gauss' law constraint imposes

$$\sigma_i^z = 1 - 2 \sum_\sigma n_{i\sigma} + 4 n_{i\uparrow} n_{i\downarrow}$$

- Introduce gauge-invariant fermion operators as before: $\tilde{c}_{i\sigma}^{(\dagger)} = \sigma_i^x c_{i\sigma}^{(\dagger)}$

Spinful fermions

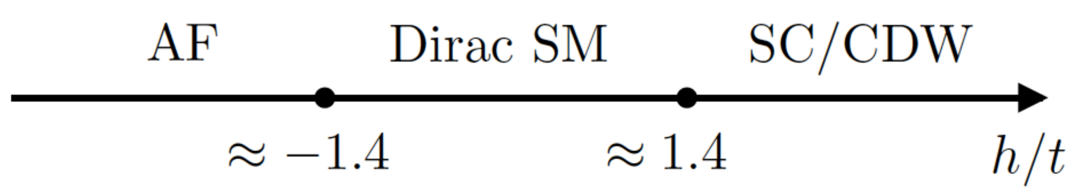
- Maps to **Hubbard model** in a **static** Z_2 gauge field:

$$\tilde{H} = -t \sum_{\langle ij \rangle, \sigma} B_{ij} \tilde{c}^{\dagger}_{i\sigma} \tilde{c}_{j\sigma} - \mu \sum_{i\sigma} \tilde{n}_{i\sigma} - 4h \sum_{i} \left(\tilde{n}_{i\uparrow} - \tfrac{1}{2}\right) \left(\tilde{n}_{i\downarrow} - \tfrac{1}{2}\right)$$

- Z_2 gauge field mediates short-range (on-site) interaction between spin up/down fermions

- Half-filling: Lieb's theorem implies π flux per plaquette for all *h*

- π-flux Hubbard model is sign-problem-free (Otsuka, Hatsugai, PRB '02; Parisen Toldin, Hohenadler, Assaad, Herbut, PRB '15; Otsuka, Yunoki, Sorella, PRX '16):



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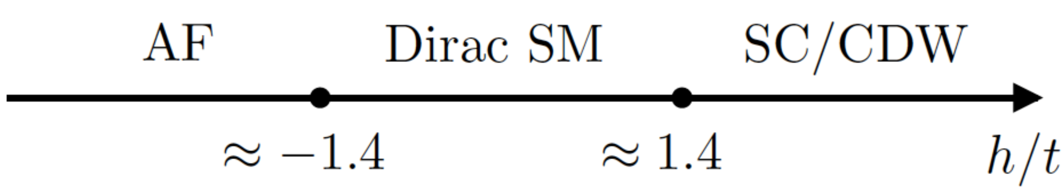
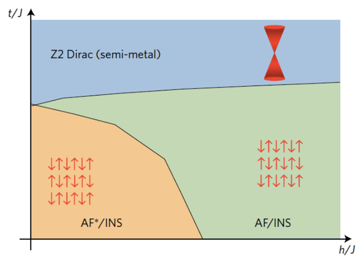
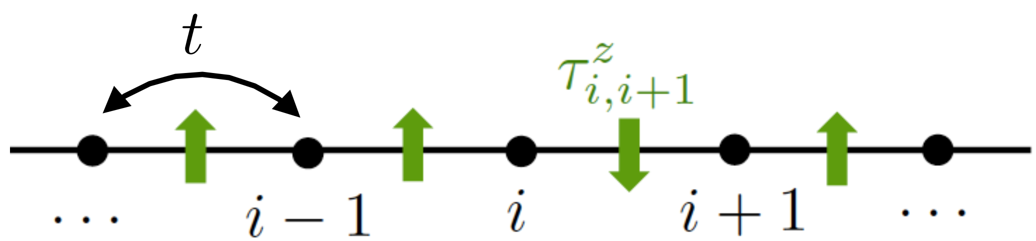




Z_2 gauge theories in (1+1)D

- Similar models can be considered in (1+1)D:



$$H_g = -h \sum_i \tau^x_{i,i+1} \longrightarrow H_g = -h \sum_i \tau^x_{i-1,i}\tau^x_{i,i+1}$$

$\sim \int dx\, E_x^2$ confining $\sim \int dx\, (\partial_x E_x)^2$ deconfined Z_2 charges in (1+1)D!

- Disorder variables: $\sigma^z_i = \tau^x_{i-1,i}\tau^x_{i,i+1}, \qquad \sigma^x_i = \prod_{j<i} \tau^z_{j,j+1}$

(Fradkin, Susskind, PRD '78)

- Gauged spinless fermions → free gauge-invariant fermions
- Gauged spinful fermions → 1D Hubbard model for gauge-invariant fermions (half filling: charge/spin gap, away from half filling: Luttinger liquid)

Unconstrained gauge theories

- Keep all Z_2 background charge sectors in the Hilbert space (e.g., Assaad, Grover, PRX '16)
- Models with $(Z_2)^N$ symmetry, N = # of sites: intermediate between constrained Z_2 gauge theory and theory with Z_2 global symmetry (e.g., fermions coupled to Ising order parameter: Schattner, Lederer, Kivelson, Berg, PRX '16; Xu et al., PRB '17)
- Consider two examples: **spinless fermions** and **Majorana fermions**
- Introduce disorder variables and gauge-invariant fermions, but **without** projecting to gauge invariant subspace
- Equivalent to **slave-spin representation** of **Falicov-Kimball**-type models for gauge-invariant fermions

Falicov-Kimball model

- Model of itinerant $\tilde{c}$ fermions interacting with localized $\tilde{f}$ fermions (Falicov, Kimball, PRL '69):

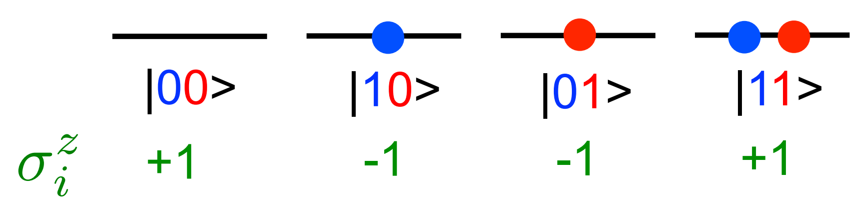
$$\tilde{H} = -t \sum_{\langle ij \rangle} B_{ij} \tilde{c}_i^\dagger \tilde{c}_j - \mu_{\tilde{c}} \sum_i \tilde{c}_i^\dagger \tilde{c}_i - \mu_{\tilde{f}} \sum_i \tilde{f}_i^\dagger \tilde{f}_i + \tilde{U} \sum_i \tilde{c}_i^\dagger \tilde{c}_i \tilde{f}_i^\dagger \tilde{f}_i$$

- Introduce **slave-spin representation** = Z_2 version of the U(1) slave-rotor representation: fractionalize fermion into Ising spin and slave-fermion (Huber, Rüegg, PRL '09; Nandkishore, Metlitski, Senthil, PRB '12)

$$\tilde{c}_i^{(\dagger)} = \sigma_i^x c_i^{(\dagger)}, \qquad \tilde{f}_i^{(\dagger)} = \sigma_i^x f_i^{(\dagger)}$$

- Fractionalization imposes local Z_2 constraint (σ^z = mod 2 occupation):

$$(-1)^{c_i^\dagger c_i} \sigma_i^z = 1 - 2 f_i^\dagger f_i$$



Falicov-Kimball model

- (2+1)D Z_2 gauge theory with spinless fermions but **without** projection to the gauge invariant subspace is **equivalent** to **2D Falicov-Kimball model in a static Z_2 gauge field**

$$\tilde{H} = -t \sum_{\langle ij \rangle} B_{ij} \tilde{c}_i^\dagger \tilde{c}_j - \mu_{\tilde{c}} \sum_i \tilde{c}_i^\dagger \tilde{c}_i - \mu_{\tilde{f}} \sum_i \tilde{f}_i^\dagger \tilde{f}_i + \tilde{U} \sum_i \tilde{c}_i^\dagger \tilde{c}_i \tilde{f}_i^\dagger \tilde{f}_i$$

$$\Updownarrow$$

$$H = -t \sum_{\langle ij \rangle} B_{ij} \sigma_i^x \sigma_j^x c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i - h \sum_i \sigma_i^z$$

- Background Z_2 charge configurations = $\tilde{f}$ electron configurations

- For μ = 0, ground state flux configuration is π **flux per plaquette** (Lieb); background Z_2 charges form **(π,π) crystalline order** (Kennedy, Lieb, Physica A '86): $\tilde{c}$ electrons acquire **massive Dirac spectrum** (~ Semenoff mass)

Majorana-Falicov-Kimball model

- Consider Falicov-Kimball model with a *p*-wave pairing term:

$$\tilde{H} = -\frac{it}{2}\sum_{\langle ij\rangle} B_{ij}\underline{(\tilde{c}_i^\dagger + \tilde{c}_i)(\tilde{c}_j^\dagger + \tilde{c}_j)} - \mu_{\tilde{c}}\sum_i \tilde{c}_i^\dagger \tilde{c}_i - \mu_{\tilde{f}}\sum_i \tilde{f}_i^\dagger \tilde{f}_i$$
$$+\tilde{U}\sum_i \tilde{c}_i^\dagger \tilde{c}_i \tilde{f}_i^\dagger \tilde{f}_i$$

- Gapless *p*-wave $\tilde{c}$ fermion superconductor (1 dispersing Majorana band + 1 flat Majorana band) interacting with localized $\tilde{f}$ fermions

- Equivalent to gauged (but unconstrained) superconductor:

$$H = -\frac{it}{2}\sum_{\langle ij\rangle}(c_i^\dagger + c_i)\tau_{ij}^z(c_j^\dagger + c_j) - h\sum_{+}\prod_{ij\in +}\tau_{ij}^x$$

Majorana-Falicov-Kimball model

- Model can be solved **exactly**:
  - Slave-spin representation: $\tilde{c}_i^{(\dagger)} = \sigma_i^x c_i^{(\dagger)}, \qquad \tilde{f}_i^{(\dagger)} = \sigma_i^x f_i^{(\dagger)}$
  - Introduce new Majorana operators: $\Gamma_i^\alpha = (\Gamma_i^\alpha)^\dagger = \sigma_i^\alpha (c_i^\dagger + c_i),$

    $\alpha = x, y, z$
  - Slave-spin Hamiltonian becomes a **free Majorana model**:

$$H = -\frac{it}{2} \sum_{\langle ij \rangle} B_{ij} \Gamma_i^x \Gamma_j^x + ih \sum_i \Gamma_i^x \Gamma_i^y$$

  - Partition function/correlation functions can be calculated **without the local Z_2 constraint** owing to a local particle-hole symmetry:

$$Z = \mathrm{Tr}\, e^{-\beta \tilde{H}} P = \frac{1}{2^N} \mathrm{Tr}\, e^{-\beta H}$$

projector implementing local Z_2 constraint

volume of local Z_2 gauge group

Projection

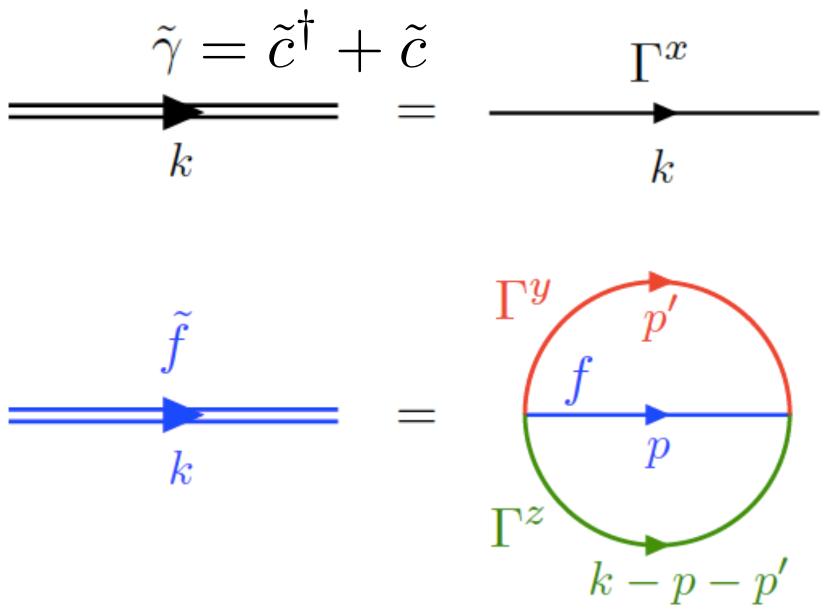
- Imagine making gauge coupling site-dependent $h \rightarrow h_i$: model has "local" particle-hole symmetry: $Z(\ldots, h_i, \ldots) = \mathrm{Tr}\, e^{-\beta \tilde{H}} = Z(\ldots, -h_i, \ldots)$

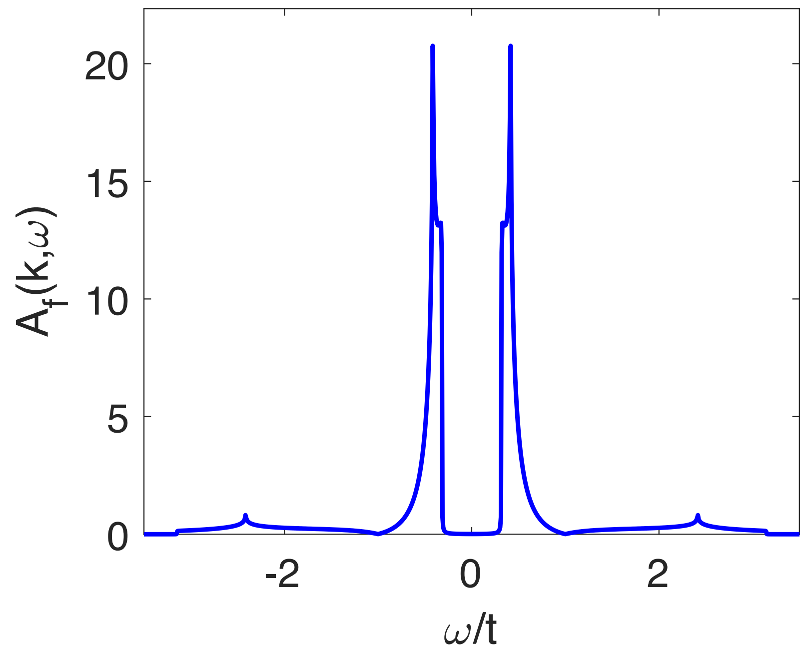
- Using $\sigma_i^x P_i \sigma_i^x = 1 - P_i$, one obtains:

$$
\begin{aligned}
Z(h_1, h_2, \ldots) = Z(-h_1, h_2, \ldots) \\
&= \mathrm{Tr}\, e^{-\beta H(-h_1, h_2, \ldots)} \prod_j P_j \\
&= \mathrm{Tr}\, e^{-\beta \sigma_1^x H(h_1, h_2, \ldots) \sigma_1^x} \prod_j P_j \\
&= \mathrm{Tr}\, \sigma_1^x e^{-\beta H(h_1, h_2, \ldots)} \sigma_1^x P_1 \prod_{j>1} P_j \\
&= \mathrm{Tr}\, e^{-\beta H(h_1, h_2, \ldots)} (1 - P_1) \prod_{j>1} P_j = \frac{1}{2} \mathrm{Tr}\, e^{-\beta H} \prod_{j>1} P_j
\end{aligned}
$$

- Repeating for $h_2, h_3, \ldots, h_N$ eliminates P altogether

Emergent fermion Green's functions





- Itinerant Majorana fermion $\tilde{\gamma}$ behaves as **free fermion** (spectral function = delta function), but gapped by FK interaction/gauge coupling
- Localized $\tilde{f}$ fermion is **not free**: spectral function ≠ delta function, gapped by FK interaction/gauge coupling (correlated insulator)

Conclusion

- Constructed Z_2 gauge theories with dynamical fermionic matter at finite density that can be solved exactly/easily in many cases (or reduce to a known problem)

- Key ingredients: "deconfining" (Kitaev) electric field term, Ising duality, Z_2 slave-spin representation

- Phenomenology: emergent massless/massive Dirac fermions, violation of Luttinger's theorem, Z_2 flux/charge crystals, free-fermion/correlated metals, insulators, and superconductors

- Ongoing work: reinstate plaquette term in gauge field Hamiltonian (full toric code), transitions out of Z_2 topological order by coupling to fermions? Also study $Z_{N>2}$ gauge theories with fermions (QMC sign problem likely)

C. Prosko, S.-P. Lee, and JM, PRB 96, 205104 (2017) [arXiv:1708.08507]